

PROJECTIVE VARIETIES COVERED BY TRIVIAL FAMILIES

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ABSTRACT. Let k be a field of characteristic zero, C a smooth projective curve defined over k . Let X, Y be projective schemes over C , with Y reduced and $g : X \rightarrow Y$ a morphism defined over C such that $g^\# : \mathcal{O}_Y \rightarrow g_*\mathcal{O}_X$ is an isomorphism. If $X \rightarrow C$ is a trivial family, then the generic fiber of $Y \rightarrow C$ is isotrivial.

1. INTRODUCTION

Let k be a field of characteristic zero, C a smooth projective curve defined over k with function field F .

Definition 1.1. Let S be a scheme over k and $\pi : X \rightarrow S$ a flat family of schemes. Then

- (a) π is called *trivial* if there exists a scheme X_0 defined over k such that $X \cong X_0 \times_k S$.
- (b) π is called *isotrivial* if there exists a finite unramified cover $S' \rightarrow S$ such that $\pi_{S'} : X \times_S S' \rightarrow S'$ is trivial.

Let X, Y be projective schemes over C , with Y reduced and $g : X \rightarrow Y$ a morphism defined over F such that $g^\# : \mathcal{O}_Y \rightarrow g_*\mathcal{O}_X$ is an isomorphism. We show that if the family $X \rightarrow C$ is trivial then the generic fiber of the family $Y \rightarrow C$ is isotrivial. (cf. Thm. 2.7)

Sketch of proof: The deformations of Y are controlled by its differentials. We study the deformations locally i.e. over a discrete valuation ring R on the curve C , and show that the fundamental exact sequence associated to the diagram $X \rightarrow Y \xrightarrow{p} \text{Spec}(R) \rightarrow \text{Spec}(k)$

$$(1) \quad 0 \rightarrow p^*\Omega_{\text{Spec}(R)/\text{Spec}(k)} \rightarrow \Omega_{Y/\text{Spec}(k)} \rightarrow \Omega_{Y/\text{Spec}(R)} \rightarrow 0$$

is split exact and consequently the deformation of Y is governed by that of X . Next we consider an infinitesimal deformation of $Y \rightarrow \text{Spec}(R)$ over the henselization of R (denoted by \tilde{R}) and show that the sequence above remains split exact at every level of the deformation. Finally we use a result of M. Greenberg to pass from \tilde{R} to R .

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We finish the article by giving an example where the result can be applied in the setting of algebraic dynamics. As an application we shall use this result in a forthcoming paper with Lucien Szpiro, on parametrization of points of canonical height zero of an algebraic dynamical system.

Notation. Throughout this paper k denotes a field of characteristic zero, C a smooth projective curve over k with function field F . Given a scheme X over C we denote its generic fiber by X_F .

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2. MAIN RESULT

Remark: In this section we assume that k is algebraically closed. The proofs work without this hypothesis with some minor modifications. For ease of notation we denote $\text{Spec}(R)$ and $\text{Spec}(k)$ by R and k respectively in the sheaves of differentials.

Proposition 2.1. *Let X, Y be projective schemes over C with Y reduced, $X \rightarrow C$ a trivial family, and $g : X \rightarrow Y$ a C -morphism such that $g^\# : \mathcal{O}_X \rightarrow g_*\mathcal{O}_Y$ is an isomorphism. Let U be a nonempty open subset of C such that Y is flat over U and for $Q \in U$, let R be the local ring at Q . Let*

$$X \xrightarrow{g} Y \xrightarrow{p} \text{Spec}(R) \rightarrow \text{Spec}(k)$$

be morphisms of schemes, then the associated sequence of differentials

$$(2) \quad p^*\Omega_{R/k} \rightarrow \Omega_{Y/k} \rightarrow \Omega_{Y/R} \rightarrow 0$$

is split exact.

Proof. $X \rightarrow C$ is a trivial family therefore the sequence of differentials on X

$$(3) \quad 0 \rightarrow g^*p^*\Omega_{R/k} \rightarrow \Omega_{X/k} \rightarrow \Omega_{X/R} \rightarrow 0$$

is split exact. Since pullbacks preserve right exactness, the sequence of differentials on Y pulled back to X along g

$$(4) \quad g^*p^*\Omega_{R/k} \rightarrow g^*\Omega_{Y/k} \rightarrow g^*\Omega_{Y/R} \rightarrow 0$$

is exact. $g : X \rightarrow Y$ induces morphisms between the above two exact sequences, yielding the following commutative diagram:

$$(5) \quad \begin{array}{ccccccc} g^*p^*\Omega_{R/k} & \longrightarrow & g^*\Omega_{Y/k} & \longrightarrow & g^*\Omega_{Y/R} & \longrightarrow & 0 \\ \downarrow q & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & g^*p^*\Omega_{R/k} & \longrightarrow & \Omega_{X/k} & \longrightarrow & \Omega_{X/R} \longrightarrow 0 \end{array}$$

We now show that q is the identity map. Since differentials contain local information it suffices to check the commutativity of the left square locally. Let $S \subset X, T \subset Y$ be open affine subsets and $A = \Gamma(S, \mathcal{O}_X)$ and $B = \Gamma(T, \mathcal{O}_Y)$. From the k -algebra homomorphisms $k \rightarrow R \rightarrow A$ and $k \rightarrow R \rightarrow B$ we get a A -module homomorphism $\alpha : \Omega_{R/k} \otimes_R A \rightarrow \Omega_{A/k}$ and a B -module homomorphism $\beta : \Omega_{R/k} \otimes_R B \rightarrow \Omega_{B/k}$. The commutativity of the left square is equivalent to commutativity of the following diagram (of A -modules):

$$\begin{array}{ccc} \Omega_{R/k} \otimes_R B \otimes_B A & \longrightarrow & \Omega_{B/k} \otimes_B A \\ \downarrow & & \downarrow \\ \Omega_{R/k} \otimes_R B \otimes_B A & \longrightarrow & \Omega_{A/k} \end{array}$$

which is equivalent to the commutativity of following triangle (of R -modules):

$$\begin{array}{ccc} \Omega_{R/k} & \longrightarrow & \Omega_{B/k} \\ & \searrow & \downarrow \\ & & \Omega_{A/k} \end{array}$$

and the commutativity of the triangle is clear. Thus q is the identity map. Since (3) is split exact and q is the identity map, we conclude that (4) is split exact. Since g_* preserves direct sums, we have

$$g_* g^* \Omega_{Y/k} \cong g_* g^* \Omega_{Y/R} \oplus g_* g^* p^* \Omega_{R/k}$$

The natural map $\Omega_{Y/k} \rightarrow g_* g^* \Omega_{Y/k}$ induces the following commutative diagram:

$$\begin{array}{ccccccc} p^* \Omega_{R/k} & \longrightarrow & \Omega_{X/k} & \longrightarrow & \Omega_{X/R} & \longrightarrow & 0 \\ \downarrow g^\# & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & g_* g^* p^* \Omega_{R/k} & \longrightarrow & g_* g^* \Omega_{Y/k} & \longrightarrow & g_* g^* \Omega_{Y/R} \longrightarrow 0 \end{array}$$

Note that the bottom row is split exact, $p^* \Omega_{R/k} \cong \mathcal{O}_Y$ and $g_* g^* p^* \Omega_{R/k} \cong g_* \mathcal{O}_X$. By assumption $g^\# : \mathcal{O}_Y \rightarrow g_* \mathcal{O}_X$ is an isomorphism thus the sequence (2) is split exact. \square

We now consider an infinitesimal deformation of Y over the henselian discrete valuation ring, denoted \tilde{R} and proceed to show that the family $Y_{\tilde{R}} \rightarrow \text{Spec}(\tilde{R})$ is isotrivial. Before we proceed we need the following definitions:

Definition 2.2. Let X, S be schemes of finite type over k and $f : X \rightarrow S$ a morphism of schemes. If f is smooth then the sequence

$$(6) \quad 0 \rightarrow f^* \Omega_S \rightarrow \Omega_X \rightarrow \Omega_{X/S} \rightarrow 0$$

is exact. This extension is non-trivial in general and is given by a class $c \in \text{Ext}^1(\Omega_{X/S}, f^*\Omega_S)$. Since $f^*\Omega_S$ is locally free, one has

$$\text{Ext}_{\mathcal{O}_X}^1(\Omega_{X/S}, f^*\Omega_S) \cong \text{Ext}_{\mathcal{O}_X}^1(\mathcal{O}_X, T_{X/S} \otimes f^*\Omega_S) \cong H^1(X, T_{X/S} \otimes f^*\Omega_S)$$

The image of c by the canonical map

$$H^1(X, T_{X/S} \otimes f^*\Omega_S) \rightarrow H^0(S, R^1 f_*(T_{X/S} \otimes f^*\Omega_S)) = H^0(S, R^1 f_* T_{X/S} \otimes \Omega_S)$$

is called the *Kodaira-Spencer class of X/S* . One can view this class as a morphism also i.e. the Kodaira-Spencer morphism

$$\kappa_{X/S} : T_S \rightarrow R^1 f_* T_{X/S}$$

The fiber $(\kappa_{X/S})_s = \kappa_s : T_{S,s} \rightarrow H^1(X_s, T_{X_s})$ is the Kodaira-Spencer map at $s \in S$.

The Kodaira-Spencer map at s measures how X_s deforms in the family X/S in the neighbourhood of s .

Definition 2.3. ([1], pp. 255) A local ring A is *henselian* if every finite A -algebra B is a product of local rings. We define the *henselization* of A to be a pair (\tilde{A}, i) , where \tilde{A} is a local henselian ring and $i : A \rightarrow \tilde{A}$ is a local homomorphism such that: for any local henselian ring B and any local homomorphism $u : A \rightarrow B$ there exists a unique local homomorphism $\tilde{u} : \tilde{A} \rightarrow B$ such that $u = \tilde{u} \circ i$.

Let $R = \mathcal{O}_{U,Q}$, the local ring at $Q \in U$, \mathfrak{m} its maximal ideal and let \tilde{R} denote the henselization of R . Define $R_n = \tilde{R}/\mathfrak{m}^{n+1}$ for each $n \geq 0$. There are natural maps $\text{Spec}(\tilde{R}) \rightarrow \text{Spec}(R)$ and $\text{Spec}(R_{n-1}) \rightarrow \text{Spec}(R_n)$ induced by the projections $R_n \rightarrow R_{n-1}$ for $n \geq 1$. Define $\tilde{Y} := Y \times_R \text{Spec}(\tilde{R})$, $Y_n := \tilde{Y} \times_R \text{Spec}(R_n)$ for each $n \geq 0$. We have the following commutative diagram of schemes (we do not write Spec in the bottom row):

$$(7) \quad \begin{array}{ccccccc} Y_F & \longrightarrow & Y & \longleftarrow & \tilde{Y} & \longleftarrow \cdots \longleftarrow & Y_n & \longleftarrow \cdots \longleftarrow & Y_0 \\ \downarrow & & \downarrow p & & \downarrow & & \downarrow p_n & & \downarrow p_0 \\ F & \longrightarrow & R & \longleftarrow & \tilde{R} & \longleftarrow \cdots \longleftarrow & R_n & \longleftarrow \cdots \longleftarrow & k \end{array}$$

Proposition 2.4. For each $n \geq 0$ the sequence of differentials associated to $Y_n \rightarrow \text{Spec}(R_n)$ i.e.

$$(8) \quad 0 \rightarrow p_n^* \Omega_{R_n/k} \rightarrow \Omega_{Y_n/k} \rightarrow \Omega_{Y_n/R_n} \rightarrow 0$$

is split exact. Moreover, $Y_n \rightarrow \text{Spec}(R_n)$ is trivial.

Proof. Pulling back (2) along the natural map $\text{Spec}(R_n) \rightarrow \text{Spec}(R)$ we get the sequence (8). Since pullbacks preserve direct sums, the sequence (8) is split exact. Hence the Kodaira-Spencer class of $Y_n/\text{Spec}(R_n)$ is trivial. In other words, $Y_n \rightarrow \text{Spec}(R_n)$ is trivial. It follows that $Y_n \cong Y_0 \times_k \text{Spec}(R_n)$ for each $n \geq 0$. \square

Definition 2.5. If V, W and T are S -schemes, an S -isomorphism from V to W parametrized by T will mean a T -isomorphism from $V \times_S T \rightarrow W \times_S T$. The set of all such isomorphisms will be denoted by $\underline{Isom}_S(V, W)(T)$.

The association $T \mapsto \underline{Isom}_S(V, W)(T)$ defines a contravariant functor

$$\underline{Isom}_S(V, W) : (S - \text{schemes})^\circ \rightarrow (\text{Sets})$$

The functor $\underline{Isom}_S(V, W)$ is representable whenever V, W are flat and projective over S . For a proof of the representability of the \underline{Isom} functor we refer the reader to ([2] pp. 132-133). We denote the scheme representing the functor $\underline{Isom}_S(V, W)$ by $Isom_S(V, W)$.

To conclude that the family $Y_F \rightarrow \text{Spec}(F)$ is trivial we need the following result of Greenberg:

Theorem 2.6. *Let \tilde{R} be a henselian discrete valuation ring, with t the generator of the maximal ideal. Let \tilde{Z} be a scheme of finite type over \tilde{R} . Then \tilde{Z} has a point in \tilde{R} if and only if \tilde{Z} has a point in \tilde{R}/t^n for every $n \geq 1$.*

Proof. [3]. □

Theorem 2.7. *Let X, Y be projective schemes over C , with Y reduced, $X \rightarrow C$ a trivial family and $g : X \rightarrow Y$ a C -morphism such that $g^\# : \mathcal{O}_Y \rightarrow g_* \mathcal{O}_X$ is an isomorphism. Then $Y_F \rightarrow \text{Spec}(F)$, the generic fiber of $Y \rightarrow C$ is isotrivial. In particular,*

$$Y_{F'} \cong Y_0 \times_k \text{Spec}(F')$$

where F' is a finite extension of F .

Proof. Observe that \tilde{Y} and $Y_0 \times_k \text{Spec}(\tilde{R})$ are flat, projective over $\text{Spec}(\tilde{R})$. Let $\underline{Isom}_{\tilde{R}}(\tilde{Y}, Y_0 \times_k \text{Spec}(\tilde{R}))(T)$ be the set of isomorphisms from

$$\tilde{Y} \times_{\tilde{R}} T \rightarrow (Y_0 \times_k \text{Spec}(\tilde{R})) \times_{\tilde{R}} T$$

Let $Z' = \text{Isom}_{\tilde{R}}(\tilde{Y}, Y_0 \times_k \text{Spec}(\tilde{R}))$ be the scheme representing the functor $\underline{Isom}_{\tilde{R}}(\tilde{Y}, Y_0 \times_k \text{Spec}(\tilde{R}))$. Note that Z' is of finite type over \tilde{R} and for $Z = \text{Isom}_{\tilde{R}}(\tilde{Y}, Y_0)$ we have $Z' = Z \times_{\tilde{R}} \text{Spec}(\tilde{R})$. By Proposition 2.4, $Y_n \cong Y_0 \times_k \text{Spec}(R_n)$ for each $n \geq 0$. Thus Z' has a R_n -point for every $n \geq 0$. By the previous theorem Z' has a \tilde{R} -point i.e. $\tilde{Y} \cong Y_0 \times_k \text{Spec}(\tilde{R})$. Note that $\text{Spec}(\tilde{R}) = \varprojlim \text{Spec}(R_n)$ and $Z'(\text{Spec}(\tilde{R})) = \varprojlim Z(\text{Spec}(R_n))$. Hence there exists an étale cover R' of R such that $Y_{R'} \cong Y_0 \times_k \text{Spec}(R')$. Thus F' (the quotient field of R') satisfies the requirements of the theorem. □

In summary, we have shown that if X, Y are projective schemes over C with Y reduced, $X \rightarrow C$ a trivial family and $g : X \rightarrow Y$ satisfies $\mathcal{O}_Y \cong g_* \mathcal{O}_X$, then $Y_F \rightarrow \text{Spec}(F)$ is isotrivial. Now we extend this result in the realm of algebraic dynamics.

3. FURTHER QUESTIONS

Let k, C, F be as in the Introduction and let C' be a finite unramified extension of C defined over the field k' .

Definition 3.1. A *dynamical system* is a pair (X, ϕ) where X is a projective variety and $\phi : X \rightarrow X$ is a non-constant morphism.

Definition 3.2. Let (X, ϕ) be a dynamical system defined over C i.e. X and ϕ are defined over C . We say the pair (X, ϕ) is *trivial* if there exists a dynamical system (X_0, ϕ_0) defined over k such that $X \cong X_0 \times_k C$ and $\phi = \phi_0 \times_k Id_C$. (X, ϕ) is *isotrivial* if it is trivial after base change to C' .

Question. Let (X, ϕ) and (Y, ψ) be dynamical systems defined over C and $g : X \rightarrow Y$ a morphism defined over C such that $g \circ \phi = \psi \circ g$. If (X, ϕ) is trivial, does it imply that (Y, ψ) is isotrivial?

The following example shows that one cannot answer the above question in the affirmative without additional assumptions.

Example. Let X be an abelian variety defined over $k, \phi = [2]$, the doubling morphism on X and $Y = X$. Let F be a field such that $tr.deg_k(F) = 1$, $P \in X(F)$ and τ_P be the translation by P on X . Let $g = \tau_P : X \rightarrow X$. We have the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{[2]} & X \\ \tau_P \downarrow & & \downarrow \tau_P \\ X & \xrightarrow{\psi} & X \end{array}$$

Since τ_P is an automorphism, we can define $\psi := \tau_P \circ [2] \circ \tau_P^{-1}$. For $Q \in X(k)$, $\psi(Q) = 2Q - P$. Thus ψ cannot be defined over any finite extension of k .

This example illustrates that even though Y is defined over k it is not necessary that ψ will be defined over some finite extension of k .

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